Towards a unified measurement of quality for mixed–elements

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Abstract

In this technical report, we introduce a new quality measure meant to be used in mixed–element meshes. In particular, we focus on tetrahedra, hexahedra, prisms (wedges) and pyramids. This measurement is based on the Jacobian, which is largely used in the context of Finite Element (FE) analysis, to discard invalid meshes and also, to compare meshes. The proposed measure is contrasted with other well known quality measures. Finally we propose an alternative to extend this measure to any type of polyhedron.

1 Introduction

The measurement of quality has two purposes:

- to compare the mesh; either with other meshes produced for the same domain or with the own notion of a "theoretically perfect" mesh.
- to improve it, i.e., acquire a "better" mesh starting from a poor quality state, by the use of mesh repairing algorithms.

In both cases we need to count with the notion of a "good" element. This task has been performed for the most common elements in meshes: the tetrahedron and the hexahedron. However, for other type of elements this still remains as an open problem. In this report we will introduce a new metric based on the Scaled Jacobian (J_S), that will serve as a measurement of the quality of mixed–elements . To this purpose, let us start with one of the most common and accepted notions of quality for the Hexahedron: the Jacobian.

2 Hexahedron quality: Jacobian based.

The Jacobian value of a node can be seen as a measure of "distortion" for a given node with respect to its neighbors in an element. By it self, this value is not useful as a measurement of quality because it will vary regarding the distance to other nodes in the element. For instance, consider a regular hexahedron and a scaled version of it: they will not have the same Jacobian value at their respective nodes. So they must be normalized somehow.

One approach to perform this task for the hexahedron is the Scaled Jacobian (J_S), and it has been employed by many authors [3, 4, 7, 1, 2, 8, 6]. Here we say that the J_S of node i, i.e., J_S^i is the Jacobian of the tetrahedron formed by i and the three connected nodes to it in the element; this "Jacobian" must be computed using the normalized vectors from i.

For example, consider the hexahedron of Figure 1(a) with nodes $0, \ldots, 7$ that will be labeled $n_0, \ldots, n_7$ respectively. If we are computing the Scaled Jacobian for node a, i.e., J_S^a, we define $d_i = n_i - n_a$ and $\hat{d}_i = d_i/||d_i||$. Now we can establish that:

$$\begin{aligned} J_S^0 &= \hat{d}_4 \cdot (\hat{d}_1 \times \hat{d}_3) = \hat{d}_1 \cdot (\hat{d}_3 \times \hat{d}_4) = \hat{d}_3 \cdot (\hat{d}_4 \times \hat{d}_1) = 1 \\ &\neq \hat{d}_4 \cdot (\hat{d}_3 \times \hat{d}_1) = -1 \end{aligned}$$

With this formulation it is clear that $J_S \in [-1, 1]$, and only positive values are acceptable for a FE simulation. Note that when computing J_R^n, the order of the normalized vectors in the formula: $\hat{d}_i \cdot (\hat{d}_j \times \hat{d}_k)$ must follow the right hand rule in the reference system. Otherwise, you will obtain negative values when the element is actually valid.

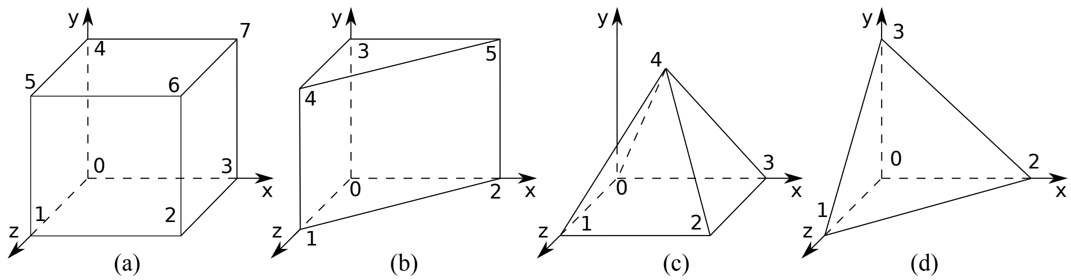

Figure 1: Basic elements: (a) hexahedron, (b) prism (wedge), (c) pyramid and (d) tetrahedron.

It is very important to keep in mind that J_S is a measurement of quality based on the Jacobian; it is not the Jacobian itself. The importance of these metrics is to establish when an element is invalid (< 0), when it is reasonably “good” ($0.2 <$) and when it is perfect ($= 1$).

3 Tetrahedron quality: Aspect Ratio Gamma.

The Scaled Jacobian (J_S) is not an accurate way of computing the quality of a tetrahedron. We have already established that the element is perfect in our scale when it takes the value $+1$. If we compute J_S for an equilateral tetrahedron, we will obtain $J_S^i = \sqrt{2}/2 \approx 0.7, \forall i = \{0, 1, 2, 3\}$, however we would expect the value $+1$ for each node. Moreover, following this definition the perfect tetrahedron does not exist, due to the fact that the quality of the element is the worst J_S in the element. Even though a tetrahedron node may reach $J_S = 1$, it is impossible that all nodes reach this threshold at the same time. And this is as it should be, because the metric is meant to be used with hexahedra.

For this reason, another type of metric is used when measuring the quality of a tetrahedron. One particular measure employed by several softwares and libraries like CUBIT, CSIMSOFT, SCIRUN, VERDICT, among others, is the so called: Aspect Ratio Gamma (ARG); described in [5]. The ARG, when computed with signed volume, detects the most common cases of invalid and poor quality tetrahedra: flat, needle and inversion (inside out element). Examples of these types of tetrahedra are shown in Figure 2. The ARG is defined as follows:

$$R = \left(\frac{1}{6} \sum_{i=0}^{5} ||l_i||^2 \right)^{\frac{1}{2}} \qquad \Rightarrow \qquad \text{ARG} = \frac{R^3\sqrt{2}}{12 \cdot V} = \frac{R^3}{8.48528 \cdot V}$$

Where $l_i, \forall i = \{0, \ldots, 5\}$ are its edge lengths and V is the volume of the tetrahedron. Now we have to put this metric in our reference scale because $\text{ARG} \in [1, \infty)$. First, because we consider signed volume, this section of the equation will detect inverted elements. Now if we define the Scaled Aspect Ratio Gamma as $\text{ARG}_S = \text{ARG}^{-1}$, we see that $\text{ARG}_S \in [-1, 0] \cup [0, 1] = [-1, 1]$, which is coherent with the scale defined for the J_S.

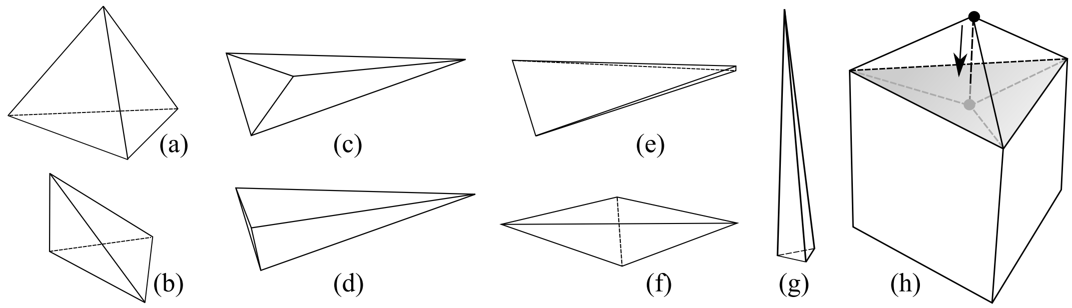

Figure 2: Different tetrahedra: (a) equilateral, (b) rectangle, (c) flat 1, (d) flat 2, (e) wedge, (f) sliver, (g) needle and (h) how an inverted tetrahedron can be formed: due to node displacement, the apex may pass through the triangular base, which will lead to a negative volume.

A strategy for using something equivalent to J_S for tetrahedra would be to normalize its value, and make it congruent with the ARG_S. This is, it should be 1 when the tetrahedron is equilateral, close to 0 when we are in presence of needle type or any type of flat tetrahedron, and negative when the element is invalid (see Figure 2(h)). Let us define the Element Normalized Scaled Jacobian (J_{ENS}) as:

$$J_{ENS} = \begin{cases} (1+k^T) - J_S & \text{if } J_S > k^T \\ J_S/k^T & \text{if } J_S \leq k^T \\ -(1+k^T) - J_S & \text{if } J_S < -k^T \end{cases}, \quad k^T = \frac{\sqrt{2}}{2} = J_S \text{ of equilateral tetrahedron.} \tag{1}$$

The reason to choose $k^T = \sqrt{2}/2$, is that this is the value of J_S at each node of an equilateral tetrahedron. We use J_S/k^T because this allows us to obtain $J_{ENS} \approx 0$ in presence of a flat or a needle tetrahedron. Let us recall that $J_S \in [-1, 1]$. It is for this reason that we have to define as special cases, when $J_S > k^T$ or $J_S < -k^T$, so the definition of J_{ENS} is congruent with the range employed by J_S. This definition is particularly useful for methods of quality improvement. Here it is important that the optimal value (+1), is reached when the node is at an optimal position.

In Table 1, values from the three quality metrics are contrasted for the set of tetrahedra shown in Figure 2: the ARG_S is computed for each element, and then the minimum and maximum value for J_S and J_{ENS} are also shown for each element. It is clear that the minimum value for these metrics are in direct correlation with ARG_S.

		Equi.	Rect.	Flat1	Flat2	Wedge	Sliver	Needle
ARG_S		1	0.769	0.004	0.004	0.003	0.005	0.021
J_S	min	0.707	0.5	0.003	0.002	0.002	0.002	0.005
	max	0.707	1	0.008	0.008	0.866	0.002	0.57
J_{ENS}	min	1	0.707	0.003	0.003	0.002	0.003	0.007
	max	1	0.707	0.012	0.011	0.841	0.003	0.806

Table 1: comparison of different quality metrics for valid tetrahedra

Now, to have more information on the behavior of these metrics, all the tetrahedra of Figure 2 were built with a face over plane XZ following the reference system of Figure 1(d). If the apex node (the other node that is not over XZ plane) is at distance d of plane XZ, we can create 7 new tetrahedra putting the apex at distance $-d$ and re–compute these values. Results are shown in Table 2.

We can see that, different than the case with valid tetrahedra, the maximum value of J_S and J_{ENS} are closer to the value of ARG_S.

Therefore, we say that the J_S of a tetrahedron is the minimum value of J_S at a node, when they are all positive and the maximum value, when they are all negative. With this definition we are closer to the values

		Equi.	Rect.	Flat1	Flat2	Wedge	Sliver	Needle
ARG$_S$		-1	-0.769	-0.004	-0.004	-0.003	-0.005	-0.021
J_S	min	-0.707	-1	-0.008	-0.008	-0.866	-0.002	-0.462
	max	-0.707	-0.5	-0.003	-0.002	-0.002	-0.002	-0.005
J_{ENS}	min	-1	-0.707	-0.012	-0.005	-0.841	-0.003	-0.653
	max	-1	-0.707	-0.004	-0.002	-0.002	-0.003	-0.007

Table 2: comparison of different quality metrics for inverted tetrahedra

of ARG_S in all possible cases. Note that if we take the minimum value all the time, for the needle and wedge tetrahedra we would have negative values away from 0. In contrast, we can now say that if the quality of a tetrahedron is close to 0, it is very close to become valid or invalid regarding its current sign, while if they have values close to -1 or $+1$, it is very unlikely that with a small displacement of a node the element will change its condition.
One last thing to mention regarding tetrahedra quality is that, ARG and ARG$_S$ are computed for the element, while J_S and J_{ENS} are computed at the nodes. This is important, once more, for repairing meshing techniques, that commonly start improving the quality by analysing poor quality nodes. Note that quality may not be improved by just analysing the worst node. For instance all the nodes of the flat tetrahedra 1 and 2, and the sliver are similar (see Figure 2), while the wedge tetrahedron has two values close to 1 and the other two close to 0 for the J_{ENS}. The needle has only one poor quality node and in this case, displacing the worst node would drastically increase the quality of the element. Finally, note that for inverted tetrahedra all values are negative for J_S and J_{ENS}, in other words, the values at the nodes may vary in magnitude, but not in sign.

4 Pyramid and Prism quality.

With respect to other types of elements, different from the tetrahedron and the hexahedron, there is little to say in terms of quality measurement. Even though there are some geometry–based quality metrics, it would be necessary to have a metric that took into account the fact that these elements must coexist among tetrahedra and hexahedra. For this reason, here we propose an extension of J_{ENS} for the pyramid and the prism (wedge). In general terms, all triangular faces should be equilaterals. With this statement we ensure that if on the other side there is a tetrahedron, it will have the option to reach "perfection". Following this, every quadrilateral should be rectangle, so a hexahedron may reach perfection too. Note that for J_S, square boxes of any shape are considered as perfect elements.
An equilateral pyramid is shown in Figure 3(a). When measuring the quality of this element, we obtain $J_S^i = \frac{\sqrt{2}}{2}\ \forall i = 0, \ldots, 4$. Now, to measure the quality of the top node t we compute J_S^t for the 4 possible tetrahedra formed by t and its neighbors in the base face. We assign to J_S^t the worst of the four possible values. Let us recall we are not computing the Jacobian, but defining a quality metric that allow us to distinguish invalid, poor quality and good elements.
We define the J_{ENS} for the pyramid exactly as for the tetrahedron, with the same constant $k^P = k^T$. Now we can show the quality for the different pyramids of Figure 3(a)–(h) in Table 3.

		Equi.	Rect.	Sliver	Wedge	Flat	Arrowhead	Inv. 1	Inv. 2
J_S	min	0.707	0.288	0.003	0.001	0.001	0.002	-0.377	-0.707
	max	0.707	1	0.006	1	0.007	0.816	0.707	-0.707
J_{ENS}	min	1	0.408	0.004	0.001	0.001	0.003	-0.533	-1
	max	1	1	0.008	0.999	0.009	0.891	1	-1

Table 3: The values of J_{ENS} for the different pyramids shown in Figure 3.

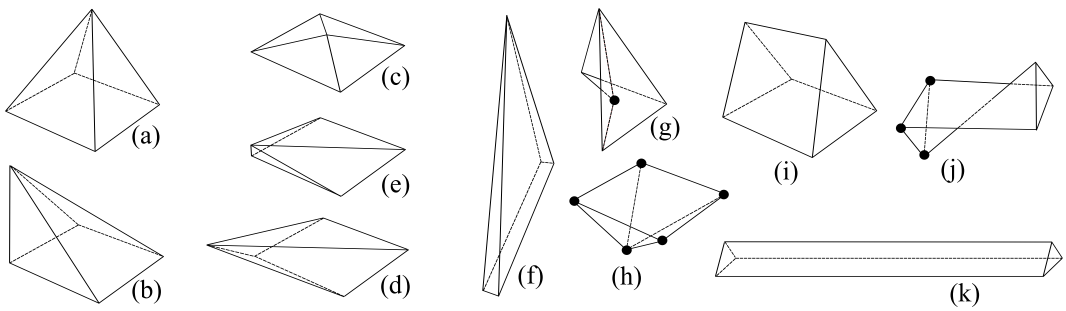

Figure 3: Different pyramid and prisms: (a) equilateral, (b) rectangle, (c) sliver, (d) wedge, (e) flat, (f) arrowhead, (g) inverted 1, on the base, (h) inverted 2, top node make all inverted, (i) referential prism, (j) inverted prism and (k) stretched prism. Marked nodes present negative quality (invalid).

The designed quality metric works as expected: it detects invalid elements, poor quality elements and it distinguishes good elements from the perfect state. However, we note that, in difference with the tetrahedron, it is possible to have some positive and negative values among the nodes. If an element has one negative node, its quality value should be the quality of the element. In order to be congruent with the metrics for tetrahedra, if the pyramid has more than one negative value, the quality of the element should be the maximum value among the negative ones. For this reason, the pyramid called Inv. 1 in the table 3 should have the value of -0.277, which is the maximum among the two negative nodes (the other one is the value shown in the table: -0.533).
Finally we repeat this procedure by creating a perfect prism: equilateral triangles and rectangular faces; then we measure the J_S and we notice that all the nodes have a value of $\sqrt{3}/2$, so once more we employ the same formula as for the tetrahedron and pyramid, only this time we change the constant $k^R = \sqrt{3}/2$. As in previous cases, this formula allows to detect invalid elements, poor quality elements and good elements with one exception. The prism shown in Figure 3(k) is a perfect element according to J_{ENS}. The reason for this is that the Scaled Jacobian (J_S) quality metric over which J_{ENS} is built, pays attention to the angles at the node and not the distances. This will also happen for the hexahedron: any hexahedron with $90°$ angles will be considered as the perfect element. Fortunately, the meshing technique described in this article does not tend to create elements with this shape. The worst ratio for an hexahedron would be equivalent to put 2 regular hexahedra next to each other. In the case of prisms, this ratio is even better.
With all the above we formally define the J^n_{ENS} quality metric for node n, and the quality of an element E_q, as follows:

$$J^n_{ENS} = \begin{cases} (1+k^e) - J_S & \text{if } J_S > k^e \\ J_S/k^e & \text{if } -k^e \le J_S \le k^e \\ -(1+k^e) - J_S & \text{if } J_S < -k^e \end{cases}$$

$$E_q = \begin{cases} \min\{J^i_{ENS}\} & \text{if } \forall J^i_{ENS} > 0 \\ \max\{J^a_{ENS}\} : J^a_{ENS} < 0 & \text{if } \exists J^i_{ENS} < 0 \end{cases}$$

Where k^e is a constant value: $k^e = k^T = k^P = \sqrt{2}/2$ for the tetrahedron and the pyramid, $k^e = k^R = \sqrt{3}/2$ for the prism and $k^e = k^H = 1$ for the hexahedron. The quality of an element (E_q) will be the minimum value of J_{ENS} at one of its nodes, if all J_{ENS} at the element are positive; and will be the maximum value of J_{ENS} among its negative nodes, when at least one is negative.
Finally, if other type of element was employed, we would need to create a version of it where all its faces should be equilateral, then measure the J_S to find its constant and employ the same formula introduced here.

5 Conclusions.

The focus of this report was to establish a quality metric to be used when the mesh has mixed-elements. We call this metric Element Normalized Scaled Jacobian J_{ENS}, which is based on the widely accepted Scaled Jacobian J_S for hexahedral meshes. In this work we adapted the J_S to measure the quality of tetrahedra and we contrasted this new metric to a widely used metric for measuring tetrahedron quality, the Aspect Ratio Gamma ARG. The values we obtained were similar when measuring poor quality tetrahedra.

Once the J_{ENS} was established for hexahedral and tetrahedral elements, it was easy to extend it to the Pyramid and Prism. If other type of element was employed, we would need to create a version of it where all its faces should be equilateral, then measure the J_S to find its constant and employ the same formula introduced at the end of section 4.

Now, one of the main advantages of having a metric that serves for different types of elements, is to be able to use it in repairing methods. Several of these algorithms use a quality metric to determine over which direction the node should be displaced in order to increase the quality of the elements attached to it. If we use different functions to measure the quality of each type of element it is sometimes not clear which direction will increase the quality of the system. This should be analysed in more details in further works.

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