DYNAMICAL BEHAVIOR OF KAUFFMAN NETWORKS
WITH AND-OR GATES

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ABSTRACT
We study the parallel dynamics of a class of Kauffman boolean nets such that each vertex
has a binary state machine \( \{ \text{AND, OR} \} \) as local transition function. We have called this
class of nets AON. In a finite, connected and undirected graph, the transient length,
attractors and its basins of attraction are completely determined in the case of only OR
(AND) functions in the net. For finite, connected and undirected AON, an exact linear
bound is given for the transient time using a Lyapunov functional. Also, a necessary and
sufficient condition is given for the diffusion problem of spreading a one all over the net,
which generalizes the primitivity notion on graphs. This condition also characterizes its
architecture. For finite, strongly connected and directed AON a non-polynomial time
bound is given for the transient time and for the period on planar graphs, together with
an example where this transient time and period are attained. Furthermore, on infinite
but finite connected, directed and non planar AON we simulate an universal two-register
machine, which allows us to exhibit universal computing capabilities.

Keywords: Discrete dynamical systems, boolean nets periods, transient times, connected
and primitive graphs, Lyapunov functionals.

1. Introduction and Definitions
Let \( G = (V, E) \) be a finite or infinite, but finite connected graph, where \( V \) is the set
of vertices, \( |V| = n \), \( E \subseteq V \times V \) is the set of edges, and \( \mathcal{V}_i \) the neighborhood of the
node \( i \), defined as: \( \mathcal{V}_i = \{ j \in V \text{ s.t. } (j, i) \in E \} \).

We will consider, in the finite case, connected or strongly connected graphs, i.e.
for every pair of vertices there exists a path connecting them. To each vertex we will
associate a state in \( \{ 0, 1 \} \) and a transition function \( f \in \{ \text{AND, OR} \} \) which depends
on the state of its neighbors, defined by \( \text{AND}(x_1, \ldots, x_p) = 1 \) if every input \( x_i = 1 \)
and \( \text{OR}(x_1, \ldots, x_p) = 1 \) if there exists at least one input \( x_i = 1 \). We will call this
kind of nets AON. We will update a configuration \( x \in \{ 0, 1 \}^n \) synchronously, which
consists into update all the vertices in a parallel way.
For the finite and undirected \( AON \) we are interested in the following problems:

1. To give sharp global bounds to the transient time, \( \tau \), which is the time the net takes to reach the steady state.
2. To characterize the steady state: maximum period, patterns of the attractors and their basins of attraction.
3. To characterize the following diffusion problem: For an initial configuration of the form: \( e_i = (0, \ldots, 1, \ldots, 0) \) to give necessary and sufficient conditions such that the net converges to the fixed point \( \tilde{1} = (1, \ldots, 1) \), i.e. to characterize the diffusion all over the net of the 1 value initially located at the vertex \( i \).

For the finite and directed \( AON \) case, we are interested into exhibit non-polynomial length transients and periods on planar graphs.

For the infinite but finite connected, directed and non-planar \( AON \) we are interested into simulate a two-register machine, which allow us to prove universal computing capabilities and to show exponential lengths of transient times and periods, see Ref. 23.

In Sec. 2 of this work we discuss the historical background and biological applications of Boolean nets. In Sec. 3 we completely characterize the dynamical behavior of a particular case of \( AON \): the finite and undirected \( OR \)-nets. A sharp bound is given for the transient time, together with a characterization of the attractors and their basin of attraction. The architecture of these nets is characterized in terms of the primitivity property. In Sec. 4 we study the general \( AON \) finite and undirected case. We give an exact linear bound for the transient time by using a Lyapunov functional. By extending the primitivity property of graphs to this class of automata, we are able to characterize the architecture of primitive \( AON \). In Sec. 5, we show that for finite and directed \( AON \), non-polynomial transient times and cycles length can be found in planar graphs. A numerical study of the two-dimensional lattice case is also performed. Finally, in section 6, we study directed and infinite but finite connected \( AON \). We show that these nets can simulate a two-register machine and therefore have universal computing capabilities.

2. Historical Background and Biological Applications of Boolean Nets

The study of boolean nets, called here Kauffman’s nets, began with Kauffman, see Refs. 20–22, who used them to model genomic regulatory systems. He proposed large nets of binary genes which interact in such a way that the output of some of them control the rate of output of others. The objective of his model was to explain essential characteristics of biological systems, like metabolic stability, cell differentiation, effects of mutations, etc., as a result of the dynamical behavior of randomly interconnected regulatory circuits of binary genes, where a state cycle is interpreted as a coordinate gene expression generating one cell type. The Kauffman classic model is a randomly constructed deterministic automata of \( n \) elements.
Its dynamics is defined by the neighborhood size $k$, which is the same for all the vertices. For each vertex its local transition function is a boolean $k$-input function, which is randomly chosen, and remains fixed in time. In Refs. 20 and 21 by numerical simulations, with $k = 2$ and 3, he determined: small cycle lengths and transient time (approximately of order $\ln(n)$); fast decreasing of the activity of nodes; the expected number of cycles per net is approximately $\sqrt{n}$, with a small average distance between cycles and high stability in front of random noise. In summary, its numerical simulations show very complicated transient trajectories but simple steady state cycles. In Ref. 22, he proves that if all the local transition functions are chosen randomly among the whole set of $k = n$ variables boolean functions with $p$ fraction of 1 results, then the expected cycle length $\bar{\tau}_{\text{max}}$ verifies:

$$\bar{\tau}_{\text{max}} \geq \left[ \frac{1}{p^2} \right]^n$$

for $n$ large enough, i.e. the expected cycle length grows exponentially with the size of the system.

Later, in Ref. 8 it was studied the pattern of the configurations that codes for the length of the steady state defined by the Kauffman’s model dynamics for each node. More recently, in Refs. 5–7, these nets were studied from a physical point of view. Derrida et al determined the following damage spreading recurrence relation, valid in the thermodynamic limit:

$$d(t + 1) = \frac{1}{2} \left[ 1 - (1 - d(t))^k \right]$$

where: $d(t) = \frac{1}{4n} \sum_{i=1}^{n} [x_i(t) - y_i(t)]^2$ is the normalized euclidean distance at time $t$ between two trajectories $x(t) = (x_i(t))_{i=1}^{n}$ and $y(t) = (y_i(t))_{i=1}^{n}$ defined by the Kauffman model dynamics. If $k \leq 2$:

$$d(t) \rightarrow 0 \quad \forall \ x(0) , \ y(0) \in \{0,1\}^n$$

i.e. the system is not chaotic; while for $k > 2$ it holds that:

$$d(t) \rightarrow d^* \quad \forall \ x(0) , \ y(0) \in \{0,1\}^n$$

where $d^*$ is the fixed point solution of equation (2), i.e. the system is chaotic. In finite dimensions on regular lattices similar results has been proven, see Refs. 3, 7 and 31.

In Refs. 25 and 26, Stauffer studied the relation between the Kauffman model and percolation phenomena. He determined numerically the existence of a percolation cluster at $p \approx 0.26$ for $k = 4$.

In Refs. 27–30, it was studied a class of boolean nets as qualitative models for genetic regulatory systems which include memorization processes. Such systems, called Kinetic Logic, consist of two kind of binary variables; the first kind, $x_i$,
represents the state of the gene \(i\) (on/off); the second kind, \(y_i\), represents the presence or absence of the product of a gene, or the concentration of gene \(i\). These variables are dynamically related through boolean functions by:

\[
y_i(t) = f_i(x_1(t), \ldots, x_n(t); y_1(t - \Delta t), \ldots, y_m(t - \Delta t))
\]  

(3)

A memorization or internal variable is defined in order to transform the last equation into a combinatorial relation. If \(z_i(t) = y_i(t - \Delta t)\) then:

\[
y_i(t) = f_i(x_1(t), \ldots, x_n(t); z_1(t), \ldots, z_m(t))
\]  

(4)

Some concrete examples of these systems can be found in Ref. 28. The regulation between genes is generated by oriented circuits of interaction, called feedback circuits or loops [27,30]. A circuit with an even number of negative interactions is called positive, since each element of the circuit exerts an activatory effect on its own evolution. On the other hand, a circuit with an odd number of negative interactions is called negative since the inhibitory effect generated on each circuit element. It has been proposed, see Refs. 27, 29 and 30 that positive circuits generate multistationarity (large number and types of attractors), that can explain cell differentiation, while negative circuits generate stable behavior (homeostatic). The role of positive circuits has also been studied in the context of memory mechanisms, such storage and evocation, see Ref. 4. In this context, our results concerning the dynamical behavior of \(AON\), can be used as an alternative explanation for multistationarity and homeostatic in terms of circuits with an even or odd number of \(AND-OR\) logical gates.

Neural Networks are well known class of boolean networks introduced to model the cognitive process, see Refs. 1, 18 and 24. Mainly, they are applied as learning and generalization algorithms for pattern recognition [1,24]. In a simple version, a neural network correspond to a discrete dynamical system defined by: a connectivity matrix \(A = (a_{ij})_{i,j=1}^n\) which represents the interaction weight between neuron \(j\) and \(i\), a threshold vector \(b = (b_i)_{i=1}^n\) and its local transition function:

\[
x_i(t+1) = \mathbb{1} \left( \sum_{j=1}^n a_{ij} x_j(t) - b_i \right)
\]  

(5)

where \(\mathbb{1}(u) = 1\) if \(u \geq 0\) and 0 otherwise. From the theoretical point of view some preliminary results were obtained in [11] for a well known class of \(OR\) nets and in a general symmetric framework of neural networks (where \(A\) is symmetric). It has been proven that the sequential iteration converges always to fixed points when \(a_{ii} \geq 0\), see Refs. 12 and 19, and only with the symmetric hypothesis, to fixed points or two cycles in the parallel update case, see Ref. 13. Furthermore, the transient time can be bounded in both cases for general symmetric matrices in \(O(||A||_1 + ||b||_1)\), where \(||A||_1 = \sum_{i,j=1}^n |a_{ij}|, \ ||b||_1 = \sum_{i=1}^n |b_i|\), see Refs. 9, 14 and 16.
Clearly, for finite and undirected $AON$ the previous results hold and since $A$ is a $\{0,1\}$-entries matrix the length of the transient time is bounded in $O(n^2)$, which also holds for matrices with coefficients bounded by a value independent of $n$. We will see that it is not a sharp bound in this case. Previous results were obtained by using Lyapunov functionals, which correspond to strictly decreasing functions with the dynamics of the automata, see Refs. 9, 12–14 and 16.

The finite and undirected $AON$ are particular cases of symmetric neural networks. It suffices to remark that:

$$\text{AND}(x_1, \ldots, x_p) = \mathbb{I}\left( \sum_{j=1}^{p} x_j - p + \frac{1}{2} \right)$$

$$\text{OR}(x_1, \ldots, x_p) = \mathbb{I}\left( \sum_{j=1}^{p} x_j - \frac{1}{2} \right)$$

Furthermore, the local transition functions $f_i \in \{\text{AND, OR}\}$ define a monotone global transition function if we consider the partial order in $\{0,1\}^n$, which compares the correspondent components of the configurations. It is important to point out that in this work we will obtain a sharp bound for the maximal transient time on finite and undirected $AON$, improving the bound obtained for the symmetric neural networks, see Ref. 15.

In Ref. 10, the dynamical behavior of the two states majority network was studied. It is easy to show that this kind of automata corresponds to a particular symmetric neural network. In this work some specific classes of graphs were studied in order to determine the kind of attractors and their number, and the conditions under which these particular graphs converge to fixed points or two-cycles. Also, an energy function was deduced, which corresponds to an alternative way to prove the fixed point or two-cycle steady state. This energy gives a bound for the maximal transient time which is of the same order of the bound given in Refs. 9, 14 and 16.

The particular case of $AON$ into which each vertex has an OR gate, called OR-nets, as local transition function is related to the graph adjacency matrix of the net. The OR-nets dynamics can be studied by analyzing the integer powers of such matrices, see Ref. 11. The properties related to the integer powers of boolean matrices have been studied extensively. A very good survey about this topic can be found in Ref. 2.

3. The Finite and Undirected OR-nets Dynamics

Suppose that every vertex in $V$ is an or-gate. It is not difficult to see that, in this case, the transition is given by $y = Ax$, where $x, y \in \{0,1\}^n$, $A$ is the adjacency matrix, and the operations considered in the matrix product are the usual boolean ones. In this context, using a characterization described in [2], we say that an undirected, finite, connected graph $G$, is primitive iff there exists an integer $k$ called the exponent of $A$, $\exp(A)$, such that $A^k > 0$ (i.e. every entry of $A^k$ is 1). From [2] it is known that $\exp(A) \leq (2n - 2)$. For adjacency matrices the primitivity
property can be formulated as follows [2]: an adjacency matrix of a connected graph is primitive if and only if the maximum common denominator, \( \text{mcd} \), of the graph circuits lengths is 1. Since in our case the graphs considered \( G \) are connected and undirected (symmetric adjacency matrix), this property is equivalent to the existence of a circuit with odd length, a condition that holds, for instance, if \( G \) contains a loop.

Let us study the parallel iteration:

\[
x(t) = F(x(t-1)) = Ax(t-1) = A^t x(0) \quad t \geq 0.
\]  

\( \text{Proposition 1.} \) The OR-net maximum transient time is \( 2(n-1) \).

\( \text{Proof.} \) Let \( x(0) \in \{0,1\}^n \) not equal to zero boolean vector. Since the graph is undirected, if there exists \( t \geq 0 \) such that \( x_i(t) = 1 \) for some \( 1 \leq i \leq n \), then by definition of the dynamics, \( x_i(t+2) = 1 \) and this situation will repeat in time. Then, every two time steps the number of ones in the configuration increases or remains constant, with the ones on the same position. It is clear that it can happen at most \( (n-1) \) times. Then the OR-nets converge in at most \( 2(n-1) \) steps. \( \square \)

Now, we will characterize the dynamical behavior in more details.

\( \text{Lemma 1.} \) Consider the parallel update for the OR-nets, then:

(i) There exist only two fixed points: \( \vec{0}, \vec{1} \in \{0,1\}^n \).

(ii) If \( G \) is primitive then there do not exist two-cycles.

(iii) If \( G \) is non-primitive then there exists a unique two-cycle denoted by \( \{x^{01}, x^{10}\} \) such that:

\[
\forall (i,j) \in E \quad x^{01}_i + x^{10}_j = 1
\]

i.e.: \( x^{10} = \overline{x^{01}} \), where \( \overline{x} \) is the boolean negation of the vector \( x \in \{0,1\}^n \).

\( \text{Proof.} \) (i) Suppose \( x \in \{0,1\}^n \setminus \{\vec{0}, \vec{1}\} \) is a fixed point, such that there exists \( (i,j) \in E \) with \( x_i = 1 \) and \( x_j = 0 \), hence by application of the OR rule to vertex \( j \) we have \( x'_j = OR(x_k, k \in V_j) = 1 \) which is a contradiction since \( x \) is a fixed point.

(ii) If \( G \) is primitive, \( \exists \ t \leq 2n - 2 \) such that \( A^t > 0 \), see Ref. 2. Thus \( \forall x \in \{0,1\}^n \setminus \{\vec{0}\} \) we have \( A^t x = \vec{1} \).

(iii) Since \( G \) is non-primitive any circuit has even length, thus the equation (5) is well defined for two consecutive elements of this circuit. Furthermore if \( OR(x^{01}) = x^{10} \) and \( OR(x^{10}) = x^{01} \), then \( \{x^{10}, x^{01}\} \) is a two cycle. Suppose that there exists a different two cycle \( \{y, OR(y)\} \), then \( \exists (i,j) \in E \) such that \( y_i = y_j = 1 \) or \( y_i = y_j = 0 \). In the first case:

\[
\forall k \in V_i \cup V_j \cup \{i,j\} : \quad OR(y)_k = 1.
\]

Furthermore, this property remains in time and it is extended to the next neighbor and so on. Thus, in finite time \( y \) converges to the vector \( \vec{1} \) which is a fixed point of OR net. For the second case, if \( OR(y)_i + OR(y)_j \neq 0 \) the iteration never comes back.
to the state \( y_i = y_j = 0 \), thus it cannot belong to a cycle. If \( OR(y)_i = OR(y)_j = 0 \) remains in time, necessarily any vertex is at state 0, then the net is in the fixed point \( x = \bar{0} \).

**Remark 1.** The two cycle induces a partition in \( V \) into two stable sets (recall that \( U \subset V \) is stable if \( \forall i, j \in U \) \( (i, j) \notin E \) defined by the Eq. (8)).

For the next results we need the following definitions.

**Definition 1.** An elementary chain, \( e(i, j) \), is a subset of \( V \) defined by:

\[
e(i, j) = \{i_0 = i, i_1, \ldots, i_{p-1}, i_p = j\}
\]

such that \( \{(i = i_0, i_1), (i_1, i_2), \ldots, (i_{p-1}, i_p = j)\} \subseteq E \). Thus an elementary chain is a set of \( p \) different edges belonging to \( E \). In an equivalent way, an elementary chain is an ordered set of \( (p + 1) \) vertices such that there exists a path in \( G \) from \( i \) to \( j \).

**Definition 2.** The length of the chain is defined by \( \ell(e(i, j)) = p \).

**Definition 3.** We will say that the chain is odd (even) if it has odd (even) length. We will also denote the chain as:

\[
e(i, j) = i(i_1)i_1(i_2)i_2 \ldots i_p(i_{p-1}, j)j.
\]

**Definition 4.** Given \( x \in \{0, 1\}^n \) we will define its support as the set: \( \text{supp}(x) = \{i \in V \text{ st. } x_i = 1\} \).

As we have stated before the steady state of the \( OR \)-net consists only of fixed points and one two-cycle. In this context we define the basin of attraction of a fixed point or of a two-cycle as the set of configurations on \( \{0, 1\}^n \) which converges to them. If \( z \) is a fixed point we will denote its basin of attraction by \( B(z) \), otherwise, if we are considering a two-cycle \( \{u, v\} \) we will denote its basin of attraction by \( B(u, v) \).

We know from Lemma 1 that in \( OR \)-nets we have two fixed points, \( \bar{0}, \bar{1} \), and only one two-cycle \( \{x^{10}, x^{01}\} \). In Lemma 3 we will characterize its basin of attraction, but we need a previous result stated in Lemma 2.

**Lemma 2.** Let \( G \) be non-primitive and \( x(t) \in \{0, 1\}^n \) with \( t \geq 1 \), such that there exist \( i, j \) with \( i \neq j \in \text{supp}(x(t)) \) and an odd chain between \( i \) and \( j \):

\[
e(i, j) = i_0(i_0, i_1)i_1(i_1, i_2)i_2 \ldots i_p(i_{p-1}, i_p)i_p
\]

with \( i = i_0, j = i_p \). Then, there exists an odd chain between vertices \( i' \in V_i \) and \( j' \in V_j \) such that \( i', j' \in \text{supp}(x(t - 1)) \).

**Proof.** From definition of the \( OR \) rule there exist \( i' \in V_i \) and \( j' \in V_j \), such that \( x_{i'}(t-1) = x_{j'}(t-1) = 1 \). Furthermore \( i' \neq j' \). If \( i'(i', i)e(i, j)(j, j')j' \) is not an odd circuit hence \( G \) is primitive which is a contradiction. Clearly if \( i' \) or \( j' \notin \{i_1, \ldots, i_{p-1}\} \)
the new chain \( i'(i', i) e(i, j)(j, j') j' \) is odd. When \( i', j' \in \{i_1, \ldots, i_{p-1}\} \), say \( i' = i_k \), \( j' = i_s \), since \( G \) is non-primitive the circuits:

\[
i(i, i_1) i_1 \ldots i_{k-1}(i_{k-1}, i_k) i_k(i_k, i) i \text{ and } i_s(i_s, i_{s+1}) i_{s+1} \ldots i_{p-1}(i_{p-1}, i_p) i_p(i_p, i_s) i_s
\]

are both even, then the associated chains:

\[
i(i, i_1) i_1 \ldots i_{k-1}(i_{k-1}, i_k) i_k \text{ and } i_s(i_s, i_{s+1}) i_{s+1} \ldots i_{p-1}(i_{p-1}, i_p) i_p
\]

are both odd. Since \( e(i, j) \) is odd, the chain \( i_k(i_k, i_{k+1}) i_{k+1} \ldots i_{s-1}(i_{s-1}, i_s) i_s \) is necessarily odd, which proves the lemma. □

**Lemma 3.** For the parallel update of an OR-net the basins of attraction are as follows:

(i) If \( G \) is primitive: \( B(\overline{1}) = \{0, 1\}^n \setminus \{\overline{0}\} \), \( B(\overline{0}) = \{\overline{0}\} \)

(ii) If \( G \) is non-primitive:

\[
B(\overline{0}) = \{\overline{0}\}
\]

\[
B(\overline{1}) = \{x \in \{0, 1\}^n \setminus \{\overline{0}\} \mid \exists i \neq j; i, j \in \text{supp}(x) \text{ and an odd chain between } i \text{ and } j\}
\]

\[
B(\{x^{01}, x^{10}\}) = \{x \in \{0, 1\}^n \setminus \{\overline{0}\} \mid \forall i \neq j; i, j \in \text{supp}(x), \text{ any chain between } i \text{ and } j \text{ is even}\}
\]

**Proof.** The case (i) follows directly from the primitivity of \( G \). Let us prove (ii). First, it is clear that if there exists \( (i, j) \in E \) with \( x_i = x_j = 1 \) then \( x \) converges to \( \overline{1} \) in at most \( (n-2) \) steps. Moreover, if \( x \neq \overline{0} \) then \( x \in B(\overline{1}) \cup B(\{x^{01}, x^{10}\}) \).

Now, suppose that \( x(0) \neq \overline{0} \) and that there exists an odd chain between \( i_0, i_p \in \text{supp}(x) \), i.e. \( i_0(i_0, i_1) i_1 \ldots i_{p-1}(i_{p-1}, i_p) i_p \) with \( x_{i_0}(0) = x_{i_p}(0) = 1 \). Then, from application of the OR-rule, there exists \( t < p \) and \( k \in \{1, \ldots, p-1\} \) such that \( x_{i_k}(t) = x_{i_{k+1}}(t) = 1 \), hence \( x(0) \) converges to \( \overline{1} \). On the other hand, if \( x(0) \) converges to \( \overline{1} \), let \( t \) such that \( x(t) = \overline{1} \), and \( t' < t \) the first step such that there exist \( i, j \in \text{supp}(x(t')) \), \( (i, j) \in E \). Since \( (i, j) \) is odd, from previous lemma \( x(0) \) admits also an odd chain, then \( x(0) \in B(\overline{1}) \). The analysis of \( B(\{x^{01}, x^{10}\}) \) follows directly from the complementarity of both sets. □

For the transient time we have the following results.

**Lemma 4.** For any OR-net considering graphs with \( p \geq 1 \) loops, the maximal transient time \( \tau \) is at most: \( (n-p) + \max(p-1, n-p) \). Moreover, this bound is attained in a one dimensional chain with \( p \) loops.

**Proof.** Since a graph containing loops is primitive, any configuration different from \( \overline{0} \) converges to \( \overline{1} \). The transient time is bounded with the time that a 1 value reach a vertex containing a loop, which is \( (n-p) \), plus the time that this 1 value diffuses all over the net, which is bounded by \( \max(p-1, n-p) \). The bound is attained in the one dimensional chain with \( p \) loops from the initial condition shown in Fig. 1. □
Lemma 5. For any $OR$-net considering graphs without loops, the transient time $\tau$ is at most $2n - 4$. Furthermore, this bound is attained for primitive graphs. (See Fig. 2).

Proof. Let $x \in \{0, 1\}^n \setminus \{0\}$ and $i \in \text{supp}(x)$. The time on which the iteration converges to $\bar{1}$ is bounded by the time that, from $x_i = 1$, the one reaches a vertex belonging to an odd cycle, that we will denote by $e$, which is $(n - |e|)$, plus the time such that any vertex in $e$ reaches the state $1$, which is $(|e| - 1)$, since in $\frac{(|e| - 1)}{2}$ steps two connected vertices have the $1$ value and in $\frac{(|e| - 1)}{2}$ steps these ones spread to all the vertices in the loop, plus the time that each vertex outside the odd cycle reaches the $1$ state, which is at most $(n - |e|)$ steps. Then:

$$\tau(i) \leq (n - |e|) + (n - |e|) + (|e| - 1) = 2n - |e| - 1$$

for any $i \in \text{supp}(x)$. Then, the convergence time is bounded by $2n - |e| - 1$. But $|e| \geq 3$ (no-loops are considered), then:

$$\tau(G) = 2n - 4$$

Now, if $\tau(G) = 2n - 4$, from previous remarks necessarily there exists $e$, with $|e| \geq 3$. Furthermore, if there exist at least two circuits the length of the path from $i \in \text{supp}(x) \setminus e$ to $e$ is strictly smaller than $(n - |e|)$. Then $\tau(G) < 2n - |e| - 1 \leq 2n - 4$. Thus, the circuit in $G$ must be unique and of length $|e| = 3$. Finally, it is easy to see that graph $\tilde{G} = (V \setminus e, E \setminus E_e)$ is a chain which joints $e$ in a unique node. Moreover, the vector $x(0) = (1, 0, \ldots, 0)$ reaches the bound in this graph, as it is shown in Fig. 2.

If $G$ is non-primitive: First, let $x \in B(\bar{1})$, such that there exist $(i, j) \in E$ and $i, j \in \text{supp}(x)$. It is direct that $x$ converges in at most $(n - 2)$ steps to $\bar{1}$. Now, suppose $x \in B(\bar{1})$ such that $\forall(i, j) \in E$ $x_i, x_j = 0$. For $i, j \in \text{supp}(x)$, let us take an odd chain of length $2p + 1$, $p \geq 1$, between vertices $i$ and $j$. Then in at most $t \leq p$ steps there will be two neighbors vertices $i_\alpha, i_\beta$ of the chain (i.e.: $(i_\alpha, i_\beta) \in E$) with $x_{i_\alpha}(t) = x_{i_\beta}(t) = 1$. Thus, it holds that:

$$\tau(x) \leq \left\lceil \frac{n - 2}{2} \right\rceil + n - 2$$
where $\lfloor \cdot \rfloor$ denotes the larger integer closer to a number, which proves the property in the non-primitive case for $x \in B(\overline{1})$. Second, let $x \in B(\{x^{01}, x^{10}\})$. If $i \in supp(x)$ then $x_j = 0 \ \forall \ j \in V_i$, otherwise $x \in B(\overline{1})$. And this property must remain in time. Thus every vertex belonging to $supp(x)$ has all its neighbors in the 0 state. This implies, by definition of the rule, that once a vertex reaches the 1 value, it enters the 1-0 cycle. Thus the maximal transient time is bounded by the time that a 1 value has been reached for the first time for every vertex on the graph, which is at most $(n - 1)$. This bound is attained in the one dimensional chain. This fact concludes the proof of lemma.

By the duality of the OR and AND rules, i.e.: $x \lor y = (\overline{x} \land \overline{y})$, the same properties that we have deduced in this section apply for the AND case, changing the role of zeros and ones on the configurations.

4. The Finite and Undirected AON Dynamics

Let us consider now the general undirected case. First, we will prove that it converges in at most a linear number of steps in the number of nodes. This result will be stated by constructing a linear Lyapunov functional. Next, we will generalize the primitivity notion presented for OR-nets and characterize such kind of AON.

**Definition 5.** Let $G = (V, E)$ be a finite, connected and undirected graph and take the partition of $V$: $\{V(OR), V(AND)\}$ such that $j \in V(OR)$ ($j \in V(AND)$) is an OR-gate (AND-gate). Let us define $G(OR)$ and $G(AND)$ by:

$$G(OR) = (V(OR), E(OR)) \quad G(AND) = (V(AND), E(AND))$$

where $E(OR) = \{(i, j) \in E / i, j \in V(OR)\}$. In a similar way we define $E(AND)$.

It is direct that $\overline{0}, \overline{1}$ are always fixed points and, since AON are particular cases of neural networks, for any initial condition they converge to fixed point or two-cycles for the parallel update.
4.1. Maximum Transient Time

In this section we obtain a linear bound for the maximal transient time of the parallel update of a general AON by given a linear Lyapunov functional, i.e. a strictly increasing function each two-times steps of the dynamics of the AON automata.

**Theorem 1.** The operator $H_{AON} : \{0, 1\}^n \rightarrow \{0, \ldots, n\}$ defined by:

$$H_{AON}(x) = \sum_{i \in \Gamma} y_i + \sum_{i \in \Gamma^c} (1 - y_i)$$

(9)

where: $\Gamma^c = V \setminus \Gamma$ and

$$\Gamma = \{ k \in V / \left[(k \in V(OR)) \land (\exists j \in V_k \cap V(OR)) \right] \lor \left[(k \in V(AND)) \land (\exists j \in V_k \cap V(AND)) \right] \}$$

$$y_i = \begin{cases} x_i & \text{if } i \in V(OR) \\ 1 - x_i & \text{if } i \in V(AND) \end{cases}$$

(10)

is an increasing Lyapunov operator for the parallel update of a general symmetric AON, each two-time steps.

**Proof.** From Sec. 1 we know that a general AON is a particular case of a neural network. Then a quadratic Lyapunov operator can be exhibited which allows to prove its two-cycles or fixed points steady state behavior. We will show that the operator defined by Eqs. (9), (10) and (11) is strictly increasing each two time steps with the dynamics of the automata, in the transient phase, that we will denote by $\{x(t)\}_{t \geq 0}$. Suppose that for $t \geq 2$ we have $x(t - 2) \neq x(t)$, then $\exists 1 \leq i \leq n$ such that: $x_i(t - 2) \neq x_i(t)$. Assume first $i \in \Gamma$. If $i \in V(OR)$, there is only one possibility (the other one is not feasible): $x_i(t - 2) = 0$, $x_i(t) = 1$. In this case: $H_{AON}(x(t)) - H_{AON}(x(t - 2)) \geq 1$. Now, if $i \in V(AND)$, there is also only one possibility: $x_i(t - 2) = 1$, $x_i(t) = 0$. In this case: $H_{AON}(x(t)) - H_{AON}(x(t - 2)) \geq 1$. If $i \in \Gamma^c$ the proof is analogous. □

**Corollary 1.** The maximal transient time for the dynamics in a finite and undirected AON is bounded by $2(n - 1)$.

**Proof.** Each 2 times steps we have that $H_{AON}$ increases its value in at least 1, independent of the vertex $i$. Otherwise, the stationary state has been reached. First, it is direct that for $x(t) \neq 0$ we have:

$$1 \leq H_{AON}(x(t)) \leq n.$$

Further we have the recurrence:

$$H_{AON}(x(t)) \geq 1 + H_{AON}(x(t - 2))$$

So,

$$n \geq H_{AON}(x(t)) \geq \frac{t}{2} + H_{AON}(x(0)) \geq \frac{t}{2} + 1$$
Hence $t \leq 2(n - 1)$ \hfill $\square$

**Corollary 2.** The following AON, from the initial configuration which is shown, attains the bound for the transient time given in the Corollary 1.

### 4.2. The Primitive Property and the Diffusion Problem

In this section we give a necessary and sufficient condition for the diffusion problem. The result is stated for configurations with only a 1, and since by the monotony of the AON automata, it can be generalized for the rest of configurations of the hypercube. The condition characterizes the architecture of such kind of graphs.

It is direct that $\overrightarrow{0}, \overrightarrow{1}$ are always fixed points and, since the AON are particular cases of neural networks, for any initial they converge to fixed points or two-cycles in the parallel update and only to fixed points in the sequential one, see Refs. 13 and 16.

**Definition 6.** We will say that a vertex $i \in V$ is a general if the initial configuration $e_i = (0, \ldots, 1, \ldots, 0)$, with the 1 state in the $i$-th position converges to $\overrightarrow{1}$, i.e. when the vertex $i$ gives the order to fire in a finite number of steps any vertex fires. It can be seen as the diffusion of a message from vertex $i$ to every vertex of the network.

**Definition 7.** We will say that $i \in V(\text{OR})$ (respectively $V(\text{AND})$) is isolated in $G$ if $V_i \subseteq V(\text{AND})$ (resp. $V(\text{OR})$). If any $i \in V(\text{OR})$ (respectively $V(\text{AND})$) is isolated in $G$, then we will say that $G(\text{OR})$ (resp. $G(\text{AND})$) is isolated in $G$.

**Theorem 2.** (1) Let $i \in V(\text{OR})$, then: $i$ is a general iff $G(\text{OR})$ is primitive and $G(\text{AND})$ is isolated in $G$.

(2) Let $i \in V(\text{AND})$ and suppose that $\{C_\ell\}_{\ell=1}^p$ is the decomposition of $G(\text{OR})$ in its connected components, i.e. $G(\text{OR}) = \bigcup_{\ell=1}^p C_\ell$ where each $C_\ell$ is a connected component. Then: $i$ is a general iff:

\[ [V(\text{AND}) \text{ is isolated in } G] \wedge \]
\[ [(\forall \ell = 1, \ldots, p \ C_\ell \text{ is primitive}) \wedge (\exists j_\ell \in C_\ell \ / \ (i, j_\ell) \in E)] \]

To prove this theorem we need the following lemma.
Lemma 6. The dynamics of $G(OR)$ is “independent” of $G(AND)$ in the following sense: Let $i \in V(OR)$ such that $x_i(t) = 1$ for $t \geq 2$, then:

(a) If $i$ is an isolated vertex in $G$: $x_i(t') = 1 \quad \forall t' = t-2k, \quad k \geq 0$.

(b) Suppose that the condition does not hold and let $x_i(t-2) = 1$.

Proof. (a) If $i$ is isolated, suppose $x_i(t) = 1$, since $i$ is isolated and $G$ is connected there exists $j \in V(AND)$ such that $(i, j) \in E$ and $x_j(t-1) = 1$. Hence, necessarily $x_i(t-2) = 1$.

(b) Suppose that the condition does not hold and let $t \geq 2$ be the first step where this situation occurs, i.e. $x_i(t) = 1$ and $x_j(t-1) = 0$ $\forall j \in V(OR) \cap V_i$. Then, $x_i(t-2) = 0$ and $\exists j \in V(AND) \cap V_i$ such that $x_j(t-1) = 1$. But, since $(i, j) \in E$ and $x_i(t-2) = 0$ one concludes $x_j(t-1) = 0$, which is a contradiction. $\square$

Now we prove the Theorem 2.

Proof. (1) $\Rightarrow$ Suppose first that $G(AND)$ is not isolated, i.e. $\exists (k, j) \in E$ with $k, j \in V(AND)$. Then $x_k(0) = x_k(0) = 0 \Rightarrow x_k(t) = x_j(t) = 0 \forall t \geq 0$ which is a contradiction. Suppose now that $G(OR)$ is not connected, then it is composed at least of 2 connected components, say $C_1$ and $C_2$. Assume $i \in C_1$; since $G$ is connected $\exists j \in V(AND), k \in V(OR) \cap V_i, \ell \in C_2$, such that $(k, j), (j, \ell) \in E$ hence cell $j$ will never take the 1 value, which is a contradiction. Suppose that $G(OR)$ is connected but not primitive. From previous lemma the behavior of $G(OR)$ is independent of $G(AND)$, i.e. if a vertex $i \in V(OR)$ takes the state 1, a neighbor of vertex $i$ in $V(OR)$ takes also the state 1 in previous step. So, from Lemma 1, $e_i \in V\{x_0, x_1\}$ then $e_i$ does not converge to $\bar{1}$.

($\Leftarrow$) Since $G(OR)$ is connected and primitive and independent of $G(AND)$ (previous lemma) the vector $e_i$ converges to $\bar{1}_{V(OR)}$. Hence in at most one step plus and since $V(AND)$ is isolated, any $i \in V(AND)$ is connected to $V(OR)$ then vector $e_i$ converges to $\bar{1}$.

(2) Suppose $i \in V(AND)$, and $G(OR) = \bigcup_{\ell=1}^{p} C_{\ell}$

($\Rightarrow$) Clearly, $V(AND)$ must be isolated. Suppose now that there exists $m \in \{1, \ldots, p\}$ such that vertex $i$ is not connected to $C_m$. So, it is direct that vertices in $C_m$ will never take the 1 value. Suppose now that, say, $C_1$ is not primitive, since the dynamics is independent for any configuration $x$ on $C_1$, $x_{C_1}$ converges to $\bar{0}$ or to the cycle $\{x_0, x_{10}\}$. In both cases $e_i$ does not reach $\bar{1}$.

($\Leftarrow$) If conditions are fulfilled we have:

$x_i(0) = 1 \Rightarrow \forall j_\ell, \ell = 1, \ldots, p, x_{j_\ell}(1) = 1$. Since $C_\ell$’s are primitive there exists $t_0 \geq 1$ such that $x_{V(OR)}(t_0) = \bar{1}$. Since $V(AND)$ is isolated, $x_{V(AND)}(t_0 + 1) = \bar{1}$ thus $e_i$ converges to $\bar{1}$. $\square$

Corollary 3. (i) If there exists $i \in V(OR)$ which is a general, then any $j \in V(OR)$ is a general. For $V(AND)$ it is not longer truth.

(ii) If any $i \in V$ is a general with $V(OR) \neq \phi$ and $V(AND) \neq \phi$ then $G(OR)$ is primitive and $G(AND)$ is isolated.
The notion of primitivity can be generalized for \( AON \) in the following way: We will say that an \( AON \) is primitive if there exists an \( \tau_{\text{max}} \) such that \( \forall \ x \neq \vec{0} \in \{0,1\}^n \ x(t) = \vec{1} \ \forall \ t \geq \tau_{\text{max}}, \) with \( x(0) = x \) and \( \{x(t)\}_{t \geq 0} \) the trajectory defined by the \( AON \) dynamic, i.e., if every configuration different from \( \vec{0} \) converges to the \( \vec{1} \) configuration (diffuses).

**Lemma 7.** \( G \) is primitive if and only if \( e_i \) is a general \( \forall \ i = 1, \ldots, n. \)

**Proof.** Direct from the monotony of the \( AON \) global transition function. \( \square \)

In the following lemma we will characterize the architecture of an \( AON \) primitive graph.

**Lemma 8.** \( G \) is primitive if and only if: \( G(\text{AND}) \) is isolated and \( G(\text{OR}) \) is primitive.

**Remark 2.** In a primitive \( AON \) all the vertices in \( V(\text{AND}) \) must have neighbors in \( V(\text{OR}) \), which also must be primitive in the sense defined in Ref. 2.

**Remark 3.** The primitivity property in general symmetric \( AON \) can be computed in polynomial time in the number of nodes, since: the maximum number of edges is quadratic; the primitivity property of the \( G(\text{OR}) \) can be computed in polynomial time, and the connectivity property can be computed in quadratic time.

5. Finite, Directed and Planar \( AON \)

In this section we study the dynamical behavior of some planar, finite, connected and directed \( AON \). First, we study the dynamical behavior of a class of finite, connected and directed \( AON \), such that every vertex belongs to an \( OR \) or \( AND \) cycle. This class of \( AON \) can be studied using a generalized version of the Lyapunov functional constructed in Theorem 1. Next, we exhibit non-polynomial time lengths on some kind of \( AON \) oriented nets. Finally we study the dynamical behavior of the two dimensional lattice, including a numerical study of the diffusion and percolation phenomena in such class of nets.

5.1. Polynomial Transient Time

Let us suppose now that the \( AON \) dynamics is defined on directed and connected graphs, such that every \( OR \) (\( AND \)) gate belongs to an \( OR \) (\( AND \)) gates directed cycle of fixed length \( p \). In the next theorem, we characterize the dynamical behavior of such nets by associating them a Lyapunov functional built based on the operator exhibited in Theorem 1.

**Theorem 3.** Suppose that we are considering \( AON \) such that each \( OR \) (\( AND \)) gate belongs to an \( OR \) (\( AND \)) gates directed cycle of fixed length \( p \). Then, the operator \( H_p : \{0,1\}^n \rightarrow \{0, \ldots, n\} \) defined by:

\[
H_p(x) = \sum_{i \in V(OR)} x_i + \sum_{i \in V(AND)} (1 - x_i)
\]
is an increasing Lyapunov operator for the parallel update, each $p$ steps.

Suppose that for $t \geq p$ we have $x(t - p) \neq x(t)$, then $\exists 1 \leq i \leq n$ such that: $x_i(t - p) \neq x_i(t)$. Assume first $i \in V(OR)$. There is only one possibility: $x_i(t - p) = 0$, $x_i(t) = 1$, since $i \in V(OR)$ and the existence of an OR gates directed cycle of length $p$, to which $i$ belongs. In this case: $H_p(x(t)) - H_p(x(t - p)) \geq 1$.

Now, if $i \in V(AND)$, there is also only one possibility: $x_i(t - p) = 1$, $x_i(t) = 0$.

In this case: $H_p(x(t)) - H_p(x(t - p)) \geq 1$.

**Corollary 4.** The maximal transient time for the dynamic in such a class of AON is bounded by $p(n - 1)$.

*Proof.* Each $p$ times steps we have that $H_p$ increases its value in at least 1, independent of the vertex $i$, and by definition of the variables $x_i$, its maximum value is $n$. This fact gives a bound of $pn$ for the transient time. To conclude the proof, it suffices to remark that a non-trivial transient behavior occurs for configurations with at least a one.

### 5.2. Non-polynomial Transient Time

In this section we will prove that a class of directed and planar AON exhibits a non-polynomial transient time in the number of nodes. Let us consider an AON, such that each $i \in V(OR)$ ($V(AND)$) belongs to a directed cycle of $OR$ ($AND$) gates of length $p_i$.

**Theorem 4.** The operator $H_c : \{0,1\}^n \rightarrow \{0,\ldots,n\}$, defined by:

$$H_c(x) = \sum_{i \in V(OR)} x_i + \sum_{i \in V(AND)} (1 - x_i) \quad (11)$$

is an increasing Lyapunov operator for the parallel update of these AON, each $mcm(p_1,\ldots,p_n)$-time steps ($mcm$ denotes the minimum common multiple).

*Proof.* Following the same schema as the proof of Theorem 3 it can be proved that $H_p(x(t)) - H_p(x(t - mcm(p_1,\ldots,p_n))) \geq 1 \ \forall \ t \geq mcm(p_1,\ldots,p_n)$, since $mcm(p_1,\ldots,p_n)$ corresponds to the synchronization time of the net. \qed

**Corollary 5.** The maximal transient time for the dynamic in such AON is bounded by $(n - 1) \cdot mcm(p_1,\ldots,p_n)$.

*Proof.* Analogous of the proof of the corollary of Theorem 3. \qed

In the next proposition we exhibit an AON which reaches a non-polynomial bound.

**Proposition 2.** Let us consider the AON directed and strongly connected net such that every vertex of the graph belongs to at least one cycle of only OR or only AND gates. Then there exist graphs in this class, such that the maximal transient time or its length of the period is at least non-polynomial in the number of nodes of the graph.
Proof. The worst case analysis considers a net with only OR circuits connected to an unique AND gate. The circuits lengths must be chosen such that, the time in which the dynamics of the circuits produces a change in the state of the AND gate is as large as possible. Let $\tau$ denotes the maximal transient time and consider circuits of length $\ell_1 = 3$, $\ell_2 = 4$ and so on. The total number of nodes is $n \geq m^2$, where $m$ is the number of circuits. Then, by Theorem 4:

$$\tau \leq (n - 1) \cdot \text{mcm}(3, 4, 5, \ldots, m)$$

It is known that, see Ref. 17:

$$\tau \leq (n - 1) \cdot \prod_{i=1}^{\pi(m)} p_i$$

where $\pi(m)$ is the number of primes less or equal to $m$. If we define:

$$\Theta(m) = \sum_{i=1}^{\pi(m)} \log(p_i)$$

then, [17]: $\Theta(m) = \Gamma_1(\pi(m)\log(m))$ and $\pi(m) = \Gamma_2(m/\log(m))$, and since $\tau \leq (n - 1) \cdot e^{\Theta(m)}$, we have that: $\tau \leq (n - 1) \cdot e^{O(n^{1/2})}$, a fact that concludes the proof.

Let us consider the following AON which satisfies the condition stated in the Theorem 4.

It consists of $(m - 2)$ directed OR circuits of length $\ell_k = k + 2$ with $k = 1, \ldots, m - 2$, each of one is connected to an AND gate by only one node of the

\[\text{V}_{\text{AND GATE}} = \text{V(OR)}\]
Fig. 5. A directed, strongly connected and planar AON with non-polynomial periodic behavior.

circuit, which also connects all the other OR gates of the circuit. The total number of nodes of the net is:

\[ n = \sum_{k=1}^{m-2} \ell_k + 1 = \frac{1}{2} (m^2 + m - 4) \]

Clearly: \( n \geq \frac{1}{2} m^2 \) for \( m \geq 4 \). The initial condition with ones only on the OR-gates connected to the AND gate has a transient time given by:

\[ \tau = (n - 1) \cdot e^{O(n^{\frac{3}{2}})} \]

With a little modification of the last example, it can be shown that in the directed and strongly connected case there exists a non-polynomial bounded periodic behavior. Let us consider the AON of Fig. 5. Clearly, it corresponds to a strongly connected directed AON such that each vertex belongs to a cycle of only OR or only AND gates. If this net starts with ones only on the beginning of the OR cycles, it can be proved, following the method of the last proposition, that this configuration will repeat after \( e^{O(n^{\frac{3}{2}})} \) iterations, since the 2-circuit of AND gates remains in the zero state.

5.3. Diffusion and Percolation Phenomena on the 2d Lattice

In this section we study numerically the diffusion and percolation problem by doing large scale simulations on two kinds of AONs defined on a 2d oriented square lattice with OR and AND gates. The first kind, that we will call the uniform lattice AON, considers cycles of length 4 with only OR or AND gates assembled uniformly in such a way to obtain a 2d oriented lattice with periodic boundary conditions, as it can be seen in Fig. 6. The second one, that we will call random lattice AON
Fig. 6. 2-d oriented lattice with uniform architecture.

Fig. 7. Mean one’s percent for uniform lattice \( AON \) at the final state vs OR-gates percent on the lattice for \( n = 128 \) and \( 256^2 \).

is constructed by a random uniform assignation of the orientation of the arcs and \( or/and \) gates over the lattice.

For such kinds of \( AON \), we studied, numerically, the diffusion and percolation problems in the following way. For a fixed lattice size the mean one’s percent at the convergence state was computed as a function of the \( or \) gates percent in the particular lattice, averaging over many randomly constructed nets and, for each net, over many initial conditions with a 10% of ones. The lattice size considered was \( n = 128^2, 256^2 \) and \( 512^2 \); the number of different random constructed lattices was chosen equal to 1000, and the number of initial conditions was 5000 (\( n = 128^2 \)), 2000 (\( n = 256^2 \)) and 1000 for \( n = 512^2 \).

Let us present, first, the results for the uniform lattice \( AON \). In Fig. 7, it is shown the mean one’s percent at the final state as a function of the OR-gates percent on the lattice.
We conclude from our simulations that the percolation parameter, defined as the percentage of $OR$ gates that allows the diffusion process and the existence of a percolation cluster for this model is approximately $0.5 \pm 0.1$. At this value, the increasing rate of the number of ones presents a large variation.

A snapshot of the final configuration for different $OR$ gates percent, can be seen in Fig. 8. The one's clusters maintain the shape of architecture of the lattice and the percolation cluster arrives from local isolated ones clusters.

Let us discuss now the results for the random lattice $AON$. In Fig. 9 it is shown the mean one's percent at the final state as a function of the $OR$-gates percent on the lattice. The behavior in this case is completely different than for the uniform lattice case. The percolation parameter for this model is approximately $0.6 \pm 0.1$. At any value, the increasing rate is almost constant.

A snapshot of the final configuration for different $OR$ gates percent, can be seen in Fig. 10. The ones clusters strongly maintain the shape of architecture of the lattice and the percolation cluster arrives from local isolated ones clusters.

6. Universality of Directed $AON$

In this section we will proof that an infinite but finitely connected, directed and non-planar $AON$ can be constructed in order to simulate a two register Turing machine. The idea is to simulate all the logical operations, see Ref. 23. From that we may say that these nets have universal computing capabilities. First, we give the logical 1 and 0 configurations.
Fig. 9. Mean one’s percent for random lattice $AON$ at the final state v/s $OR$-gates percent on the lattice for $n = 256^2$.

Fig. 10. One’s (black dots) distribution at the final state for the 2-d oriented lattice ($n = 256^2$). Percent of $OR$-gates is increased from the left-upper to the right-lower figure.

$$X \leftrightarrow \text{true} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad X \leftrightarrow \text{false} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$ (12)

The computing space for the logical operations on the encoded logical 1 and 0, will be defined by an infinite number of $OR$ gates connected in an arbitrary, but finite way. Clearly, over this space, the configurations $1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are gliders.

**Proposition 3.** The logical *not*, *and*, or operations are given by the following subgraphs:
Fig. 11. Computing space for the logical gates.

Fig. 12. Logical not operation.

Fig. 13. Logical and operation.
Fig. 14. Logical or operation.

Fig. 15. Logical xor operation.
Proof. It can be easily check that after one step of computing, the subgraphs of Figs. 12, 13 and 14 performed the respective logical operations.

The previous result allows us to compute any boolean function by properly connected subgraphs constructed based on the above logical gates. For instance, the xor logical operation, defined by:

\[ \text{xor}(x, y) = (x \land \bar{y}) \lor (\bar{x} \land y) \]

can be computed in 4 time steps, using 18 and/or gates, connected as it is shown in Fig. 15.

**Theorem 5.** There exists an Universal AON on an infinite connected and directed graph.

Proof. As it is established in Refs. 15 and 23, from previous codification and circuits it is possible to construct an arbitrary two register machine, which proves the theorem.

7. Conclusions

In this work it was studied the dynamical behavior of the parallel iteration of graphs with AND/OR gates as local transition functions. This kind of dynamical systems correspond to a particular class of neural networks.

For the finite and undirected AON we have obtained a sharp linear bound of the transient time by constructing a linear Lyapunov functional which governs the parallel dynamics. We have also determined the patterns of the attractors, its basins of attraction and a necessary and sufficient condition for the diffusion problem.

For the finite and directed AON a Lyapunov functional has been obtained for some particular kind of nets, together with non-polynomial length transients and periods on such class of planar graphs.

For the infinite connected, directed and non-planar AON we have constructed a two-register machine based on the simulation of boolean functions. This fact proved universal computing capabilities of the dynamical system under study.

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